M. Gross - 28/5/08 - Mirror symmetry for $\mathbb{P}^n$

- Mirror symmetry for $X = \mathbb{P}^n$ (with big quantum cohomology - Barannikov)

  Parameter space for big QH^* is $H^*(\mathbb{P}^n, \mathbb{C})$ (not H^2)

- Mirror is $(\check{X}, W)$: $\check{X} = \left\{ \prod_{i=0}^{n} x_i = 1 \right\} \subset \mathbb{C}^{n+1}$

$$(\check{X} \cong (\mathbb{C}^*)^n)$$

$$W = x_0 + \dots + x_n$$

<u>Universal unfolding</u>: $W_{\vec{t}} = t_0 + (1+t_1)W + t_2 W^2 + \dots + t_n W^n$

w/ complex moduli space: $\mathcal{M}_{cx} = \operatorname{Spec} \mathbb{C}[[t_0, \dots, t_n]]$

<u>Need to compute</u>: <u>B-model of LG-mirror</u> $:=$

<u>period integrals</u> $\int_{\Delta} e^{-qW_{\vec{t}}} f \Omega$

where $\Omega = \dfrac{dx_1 \wedge \dots \wedge dx_n}{x_1 \cdots x_n}$,

q = coord. on $\mathbb{C}^*$ (ray in moduli space $\to$ LCSL)

f is a suitable regular function on $\check{X} \times \left(\mathbb{C}^*_q \cup \{\infty\}\right)$

$\Delta \subset \check{X}$ is a cycle, possibly unbounded with $\operatorname{Re}(qW_{\vec{t}}) \to -\infty$ at infinity

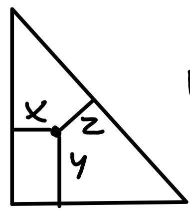
- a normalization condition tells us what f should be ($\Leftrightarrow$ in CY3 case, requiring normalization of $\int_{A_i} \Omega$ on some of the 3-cycles).

- these integrals give us canonical coordinates $y_0, \dots, y_n$ on $\mathcal{M}_{cx}$

- get Givental's J-function for $\mathbb{P}^n$ from this.

NB: W counts Maslov index 2 discs, cf. Cho-Oh

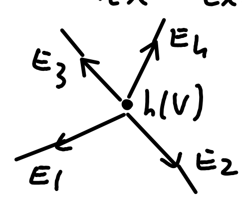
Ex: for $\mathbb{P}^2$,



$W = x + y + z$

Def: A tropical disk in $\mathbb{P}^2$ is a map $h : \Gamma \to \mathbb{R}^2$ where
- Γ graph of genus 0
- each edge E of Γ has a weight $w(E) \in \mathbb{N}$
- some edges of Γ have only one vertex (unbounded edges)
- $\exists$ unique univalent vertex
- $h(E)$ is an affine line segment (or ray if E unbounded)
- unbounded rays are parallel to $\swarrow$, $\rightarrow$, or $\uparrow$
- at any vertex except univalent, balancing condition

ie.



$\vec{v_i}$ = primitive outgoing tgt to E_i

$\sum w(E_i)\, \vec{v_i} = 0$

- Choose k points $p_1 \dots p_k$ in $\mathbb{R}^2$ and consider marked tropical disks

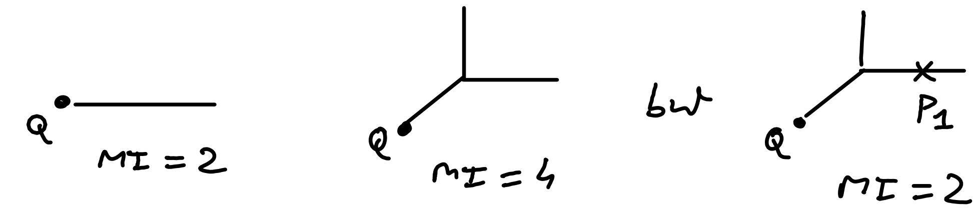
$$h : (\Gamma, x_1 \dots x_n) \to \mathbb{R}^2 \quad \text{s.t.} \quad \begin{array}{l} h(x_i) \in \{p_1 \dots p_k\} \\ h(x_i) \neq h(x_j) \end{array}$$

The Maslov index of h [actually: Maslov index in a suitable blowup] is $MI(h) := 2(\#\text{ unbounded edges of } \Gamma - n)$.

(**Idea:** count ordinary Maslov 2 discs (rigid w/out marked pts) gives usual $W = x + y + z$.

The big QH^* deformation of W involves also counting discs w/ point constraints, becoming rigid after adding constraints)

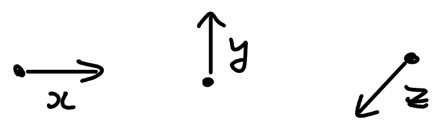
Examples



- Given a MI=2 disk h, define

$$\text{Weight}(h) = \text{Mult}(h)\, x^a y^b z^c \prod_{i\in I} u_i$$

where • a, b, c = # of unbounded edges in directions



respectively

- $\text{Mult}(h)$ = Mikhalkin multiplicity of h
- $u_1, \dots u_k$ variables associated to constraints $P_1, \dots P_k$
- $I = \{ i \in \{1..k\} / h(x_j) = P_i \text{ for some } j \}$

ie those P_i's that we do hit.

Define: $W_k(Q) = y_0 + \sum_{h} \text{Weight}(h)$ & set $xyz = e^{y_1}$

(sum over h: MI=2 disks with boundary Q) (usual relation from H^2 coordinate)

This is a deformation of W over

$$\text{Spec } \mathbb{C}[[y_0, y_1, u_1 \dots u_k]]/(u_1^2, \dots, u_k^2).$$

"Thm" (in progress)

$\exists$ map $\text{Spec } \mathbb{C}[[y_0, y_1, u_1 \dots u_k]]/(u_i^2) \longrightarrow \text{Spec } \mathbb{C}[[y_0, y_1, y_2]]$

$$y_0 \longleftarrow\!\shortmid\; y_0$$
$$y_1 \longleftarrow\!\shortmid\; y_1$$
$$\Sigma u_i \longleftarrow\!\shortmid\; y_2$$

under which $W_k \longleftarrow\!\shortmid$ univ. unfolding of W

Ie: $W_k(Q)$ as defined above should equal

$$t_0 + (1+t_1)(x+y+z) + t_2(x+y+z)^2$$

under suitable change of vars. $\left\{\begin{matrix} t_0 \\ t_1 \\ t_2 \end{matrix}\right\} \xleftrightarrow[\text{mirror map to order } k]{} \left\{\begin{matrix} y_0 \\ y_1 \\ \sum u_i \end{matrix}\right\}$

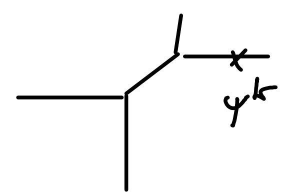
- For each k we get canonical coordinates to k^{th} order

  We should actually take a $\lim\limits_{\overleftarrow{k\to\infty}}$ ....

- Can also compute

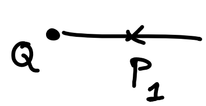
$$\int_{T^2} e^{qW_k(Q)} \frac{dx \wedge dy}{xy} = e^{qy_0}(1 + q^2 \cdots + q^3 \cdots)$$

correspond to descendent (ψ^k) GW invts



- Note: in the presence of constraint pts P_i (ie. blowups of $\mathbb{P}^2$), $\exists$ Maslov index 0 discs giving walls in space of Q's

  e.g.



$\Rightarrow$ this cuts up $\check{X}$ into chambers, separated by a scattering diagram, and need instanton corrections from these!

Rigidity $\Rightarrow$ We still have that $\check{X} \cong \{xyz = e^{y_1}\}$

but then contributions of basic discs are not quite x, y, z, rather $x(1 + O(u_i))$ ......